Exercise 16

(a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\ \mu\text{m} = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from

(i) 5 to 8 μm (ii) 5 to 6 μm (iii) 5 to 5.1 μm

(b) Find the instantaneous rate of change of V with respect to r when $r = 5\ \mu$m.

(c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).

Solution

Part (a)

The average rate of change of the volume with respect to radius is given by the slope of the secant line.

(i) $$\frac{\Delta V}{\Delta r} = m = \frac{V(8) - V(5)}{8 - 5} = \frac{\frac{4}{3}\pi(8)^3 - \frac{4}{3}\pi(5)^3}{3} = 172\pi\ \mu\text{m}^3 \text{ per micrometer of radius}$$

(ii) $$\frac{\Delta V}{\Delta r} = m = \frac{V(6) - V(5)}{6 - 5} = \frac{\frac{4}{3}\pi(6)^3 - \frac{4}{3}\pi(5)^3}{1} = \frac{364}{3}\pi \approx 121\pi\ \mu\text{m}^3 \text{ per micrometer of radius}$$

(iii) $$\frac{\Delta V}{\Delta r} = m = \frac{V(5.1) - V(5)}{5.1 - 5} = \frac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = \frac{7651}{75}\pi \approx 102\pi\ \mu\text{m}^3 \text{ per micrometer of radius}$$

Part (b)

Calculate the derivative of $V(r) = \frac{4}{3}\pi r^3$.

$$V'(r) = 4\pi r^2$$

Consequently, the instantaneous rate of change when $r = 5\ \mu$m is

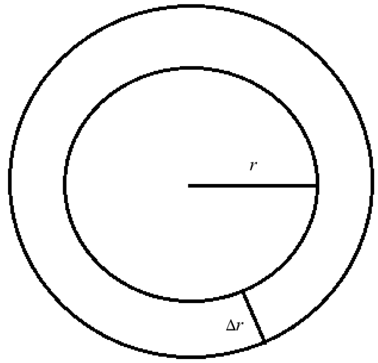
$$V'(5) = 4\pi(5)^2 = 100\pi \approx 314\ \mu\text{m}^2.$$

Part (c)

Since the surface area S of a sphere with radius r is $4\pi r^2$,

$$V'(r) = S.$$

Suppose there's a sphere with radius r, and the radius increases by Δr.



The old volume is $V_{\text{old}} = \frac{4}{3}\pi r^3$, and the new volume is

$$\begin{aligned} V_{\text{new}} &= \frac{4}{3}\pi(r + \Delta r)^3 \\ &= \frac{4}{3}\pi[r^3 + 3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3] \\ &= \frac{4}{3}\pi r^3 + 4\pi r^2\Delta r + 4\pi r(\Delta r)^2 + \frac{4}{3}\pi(\Delta r)^3. \end{aligned}$$

Because Δr is assumed to be small, $4\pi r(\Delta r)^2 + \frac{4}{3}\pi(\Delta r)^3$ is extremely small compared to $\frac{4}{3}\pi r^3 + 4\pi r^2\Delta r$ and can be neglected to a good approximation.

$$V_{\text{new}} \approx \frac{4}{3}\pi r^3 + 4\pi r^2\Delta r$$

Therefore, the approximate change in volume is

$$\begin{aligned} \Delta V &= V_{\text{new}} - V_{\text{old}} \\ &\approx \left(\frac{4}{3}\pi r^3 + 4\pi r^2\Delta r\right) - \frac{4}{3}\pi r^3 \\ &\approx 4\pi r^2\Delta r. \end{aligned}$$